Loss Increases in Multimode Rectangular Infrared Waveguides Due to Helical Deformations

MICHEL E. MARHIC, MEMBER, IEEE

Abstract—The coupling coefficients for all TE_{pq} and TM_{pq} modes of multimode rectangular waveguides, caused by elementary rotations about the three axes of symmetry, are calculated in the microwave approximation. The modes b coupled to a common mode a by these independent rotations form three nonoverlapping families. Furthermore, if a deformation consisting of no more than two of these rotations is considered, none of the modes b are coupled to each other by that deformation. These properties lead, in the case of mild deformations, to the formulation of a multimode coupling theory which shows that the loss increase due to such deformations can then be viewed as the result of multiple two-mode coupling. Explicit formulas are derived for the loss increase of TE_{p0} due to mild circular bending. Qualitative features of twists and helical deformations are also brought out.

I. Introduction

A MODE-COUPLING analysis of the bending losses in infrared metallic waveguides was recently introduced [1]. The method was used in particular to explain the high bending losses observed in practice with circular metallic waveguides at $\lambda = 10.6\ \mu m$ [2]. Some preliminary calculations were also carried out for slab waveguides, based upon the expression for the self-coupling coefficient of individual modes [3], but they were very limited because of the lack of general expressions for the coupling coefficients for all pairs of modes.

In this paper we examine in detail the change in loss for rectangular infrared metallic waveguides subjected to either circular bends or twists, by calculating all possible coupling coefficients between pairs of modes introduced by these deformations. The calculations are made under the assumption that the modes of the straight guide are the standard TE_{pq} and TM_{pq} modes of microwave theory; this is a very good approximation in the far-infrared, and appears to be fairly accurate even in the mid-infrared [1], [2]. Microwave formulas are also used for the straight guide mode losses, although there is some doubt as to the applicability of some of them in the mid-infrared [4], [5].[1] In the case of a circular y-bend, it is shown that a TE_{p0} mode couples only to $TE_{p'0}$ modes of opposite parity, and primarily to its nearest neighbors; in particular TE_{10} couples primarily to TE_{20}, as previously assumed [1]. It is also shown that the TE_{p0} $(p>1)$ loss is slightly reduced by y-bends, as proven differently by others [6], [7]. In the case of a twist, TE modes are found to couple to TE and TM modes of opposite parity; the excess loss varies quadratically with the twist rate. Finally, the case of simultaneous bending and twisting, i.e., of helical deformation, is examined.

It is found repeatedly in this study that there are important situations which cannot be studied by a simple two-mode coupling analysis, because the energy of a particular mode may be coupled to two or more other modes simultaneously, and at about the same rate. To study these situations a multimode coupling analysis is introduced, which yields a tractable formalism for small deformations. The results, however, do not account for the loss increase in the presence of substantial deformation, as encountered in the whispering gallery regime [6].

Besides providing powerful analytic tools to study the effect of deformations on multimode metallic waveguides, this work points to the fact that mode coupling is indeed intimately connected with even mild deformations of such guides, and that mode conversion in such situations cannot be dismissed on the basis of approximate criteria [8], [9], but must in fact be studied by the type of method presented here and elsewhere [1].

II. Circular Bends

Consider the rectangular guide of Fig. 1. Such a guide can be deformed slightly by a circular bend of large radius R around an arbitrary direction in the $x-y$ plane. Such general bends can, as we shall prove, be decomposed into appropriate combinations of circular bends around the x- or y-axis. We will first analyze in detail the case of a y-axis bend (hereafter referred to as a y-bend), and then deduce the results for an x-bend by analogy; the case of the general circular bend will then be examined.

The coupling coefficients between two modes a and b induced by a circular bend of radius R can be calculated in a manner analogous to that used for the self-coupling coefficient [3], [10]. Starting with a straight guide, (Fig. 2) it

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The author is with the Department of Electrical Engineering and Computer Science, Northwestern University, Evanston, IL 60201.

[1]Accurate calculations cannot be carried out for coupling to other than TE_{p0}, which are the only modes for which reliable expressions for α exist. As was shown experimentally, standard expressions derived from microwave theory cannot be used at $\lambda = 10.6\ \mu m$.

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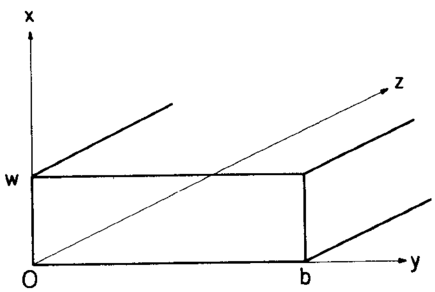

Fig. 1 Coordinate system for the rectangular guide. This choice yields the simplest expressions for the fields used to calculate the coupling coefficients. The origin will later on be moved to the center of the guide to obtain symmetric figures, a fact which will have no effect on the quantities then being discussed.

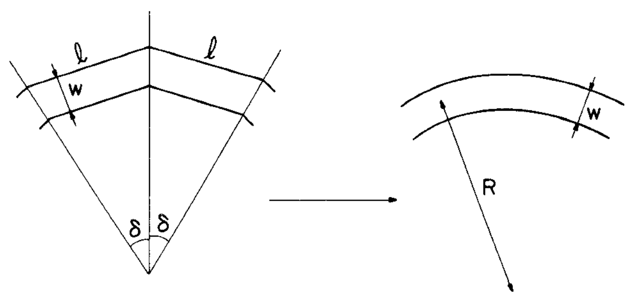

Fig 2. Circular bend as the limit of tilted straight segments

is decomposed into segments of length l, tilted with respect to each other by a small angle δ, and one then lets δ and l go to zero in such a way that

$$R = \lim_{l,\delta \to 0} (l/\delta). \tag{1}$$

In that case, if $K_{a,b}(\delta)$ is the coupling coefficient between the two modes a and b due to a single tilt of angle δ, the coupling coefficient for the same modes per unit length of the corresponding circular bend of radius R is

$$C_{a,b} = \lim_{l,\delta \to 0} \left[l^{-1} K_{a,b}(\delta) \right]. \tag{2}$$

In multimode hollow waveguides with high index walls, the coupling coefficient $K_{a,b}$ due to a discontinuity is, to a good degree of approximation, given by the overlap integral [1]

$$K_{a,b} = \frac{\iint \vec{E}'_a \cdot \vec{E}_b \, ds}{\left[\iint (E_a)^2 \, ds \right]^{1/2} \left[\iint (E_b)^2 \, ds \right]^{1/2}} \tag{3}$$

where $ds = dx\, dy$ and the integrals are calculated over the cross section of the guide after the discontinuity: $\vec{E}'_a$ is the transverse electric field distribution of mode a propagating in the first section, in the coordinate system (x, y) of the second section; $\vec{E}_a$ and $\vec{E}_b$ are the transverse field distributions of modes a and b in the second section. In the present case of the discontinuity being a small angle δ around the y-axis, we have

$$\vec{E}'_a \simeq \vec{E}_a \exp\left[i \frac{2\pi\delta x}{\lambda} \right]. \tag{4}$$

TABLE I
COEFFICIENTS $F_y C_{a,b}$ FOR CIRCULAR y-BENDS

b \ a	TE_{pq}	TM_{pq}
$TE_{p',q}$	$\frac{2p^2p'^2}{w^2} + \frac{q^2(p^2+p'^2)}{b^2}$	$\frac{pq(p'^2-p^2)}{bw}$
$TM_{p',q}$	$\frac{p'q(p^2-p'^2)}{bw}$	$pp'\left(\frac{p^2+p'^2}{w^2} + 2\frac{q^2}{b^2}\right)$

Our primary interest here is to study the effect of waveguide deformations upon the TE_{p0} modes, but we will nevertheless evaluate the coupling coefficients for all pairs of TE and TM modes. The relations necessary to carry out the calculations are given in the Appendix. Table I lists the expressions for the coupling coefficients $C_{a,b}$ multiplied by the function

$$F_y = F_{pq,p'q'}\left(\frac{w}{\lambda R}\right) = \frac{\pi}{8}(p^2 - p'^2)^2 \left(\frac{p^2}{w^2} + \frac{q^2}{b^2}\right)^{1/2} \left(\frac{p'^2}{w^2} + \frac{q'^2}{b^2}\right)^{1/2} \frac{\lambda R}{w}. \tag{5}$$

The selection rules are that a y-bend couples only modes with $q = q'$, and that p and p' must have opposite parity. Failure to satisfy these conditions leads to a vanishing coupling coefficient (to first order in R^{-1}). One exception to this rule is the case of self-coupling for which $K_{a,a} = 1 - \mathcal{O}(R^{-2})$, which must be recalculated separately. This is done with the help of (39) in the Appendix. It is found that

$$|K_{a,a}(\delta)| = 1 - \left[\frac{\Pi^2}{6} \pm \frac{1}{p^2} \frac{\left(\frac{p^2}{w^2} - \frac{q^2}{b^2}\right)}{\left(\frac{p^2}{w^2} + \frac{q^2}{b^2}\right)} \right] \left(\frac{\delta a}{\lambda}\right)^2,$$

$$-\text{for TE}, \qquad +\text{for TM} \tag{6}$$

from which it is meaningless to calculate $C_{a,a}$ from (2). This result can be related to the expression for the quantity B given in the appendix of [3], and it is found that the sign convention for TE and TM modes presented there contradicts (6). A check on the present choice of signs is obtained by verifying the consistency of (6) with the expressions of Table I. This is done by expressing that all the power incident on a tilt in a mode a must emerge from it as the sum of the powers coupled into all other modes b, including a itself. Thus we must have, to second order in δ,

$$\sum_{b \neq a} |K_{a,b}|^2 + |K_{a,a}|^2 = 1. \tag{7}$$

This relation has not been proven in general, but it has been verified numerically for mode a being TE_{10}, thus validating the choice of signs in (6).

In a rectangular metallic guide with $b \gg w$, the least attenuated mode is TE_{10}, and the whole family of TE_{p0} modes has generally lower losses than the other TE modes

TABLE II
VALUES OF $G(p,p')$

p \ p'	1	2	3	4	5	6
1		0.889	0	0.071	0	0.019
2	0.889		0.96	0	0.091	0
3	0	0.96		0.979	0	0.099
4	0.071	0	0.979		0.988	0
5	0	0.091	0	0.988		0.992
6	0.019	0	0.099	0	0.992	

and the TM modes. The TE_{p0} modes, and especially TE_{10}, are therefore preferred for low-loss transmission in such guides, and it is important to determine how their losses are affected by deformations. For a y-bend, Table I and the associated selection rules show that the TE_{p0} modes are only coupled to each other, the coupling coefficient between TE_{p0} and $TE_{p'0}$ being given by

$$C_{p0,p'0}=\frac{16}{\Pi}\frac{pp'}{(p^2-p'^2)^2}\frac{w}{\lambda R},\qquad p\neq p'. \tag{8}$$

For a given p, this coupling coefficient is largest for $p'\simeq p$, a fact put in evidence by Table II in which values of

$$C(p,p')=4\frac{pp'}{(p^2-p'^2)^2}=C_{p0,p'0}\frac{\Pi}{4}\frac{w}{\lambda R},\qquad p\neq p' \tag{9}$$

have been calculated for low values of p and p' (the value 0 is assigned when the parity rule is violated). For large n, it can be shown that

$$G(p,p\pm n)\simeq n^{-2} \tag{10}$$

so that an arbitrary TE_{p0} mode is primarily coupled to its two nearest neighbors, $TE_{p\pm1,0}$; this property even extends to low values of p, as can be seen from Table II. The situation is even simpler for TE_{10} as it is primarily coupled to a single other mode, namely TE_{20}. This finding justifies the assumption which was made elsewhere about the coupling of TE_{10} by a y-bend [1]. Having now at our disposal a more accurate expression for $C_{10,20}$, we can reevaluate the critical bending radius for y-axis bending for TE_{10} [1]; we now obtain

$$R_c=0.83\frac{w^3}{\lambda^2}. \tag{11}$$

The numerical coefficient in this expression is close to the value 1.18 which can be extrapolated from exact numerical calculations [6] or obtained by a perturbation technique [7]; it is closer than 4.6 or 0.61 arrived at by other methods ([11] and [9], respectively). The two-mode coupling approximation thus provides a satisfactory description of the additional loss suffered by the TE_{10} mode in a rectangular guide gently bent around the y-axis.

The expressions of Table I can also be used to similarly study the influence of a y-bend on the losses of higher order TE_{p0} modes ($p\geq 2$). For such a mode, however, we can no longer use the standard two-mode coupling theory [1], since we have shown that TE_{p0} couples almost identically to $TE_{p+1,0}$ and $TE_{p-1,0}$. We must, therefore, consider a three-mode coupling scheme wherein a mode of interest a is coupled to two modes b and c via the coupling coefficients $C_{a,b}$ and $C_{a,c}$, respectively; these last two modes, however, are not coupled to each other, i.e., $C_{b,c}=0$. This coupling scheme is similar to that studied by Solymar [12]. In fact, we will look at this situation as a particular case ($N=3$) of the more general N-mode coupling scheme where mode of interest 1 couples to modes $2,3,\cdots,N$, while the latter modes are not coupled to each other. In that general case the determinant equation for γ, the complex propagation constant of the eigenmodes of the deformed guide, is

$$\begin{vmatrix} \gamma-\gamma_1 & iC_{1,2} & iC_{1,3} & \cdots\cdots & iC_{1,N} \\ iC_{1,2} & \gamma-\gamma_2 & 0 \cdots\cdots & & 0 \\ iC_{1,3} & 0 & \gamma-\gamma_3 & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ iC_{1,N} & 0\cdots\cdots & & 0 & \gamma-\gamma_N \end{vmatrix}=0 \tag{12}$$

where γ_i ($i=1,\cdots,N$) is the propagation constant of the ith mode of the straight guide. Equation (12) is equivalent to

$$\left[1+(\gamma-\gamma_1)^{-1}\sum_{k=2}^{N}(C_{1,k})^2(\gamma-\gamma_k)^{-1}\right]\prod_{i=1}^{N}(\gamma-\gamma_i)=0 \tag{13}$$

or

$$\gamma-\gamma_1=\sum_{k=2}^{N}(C_{1,k})^2(\gamma_k-\gamma)^{-1}. \tag{14}$$

In the case of mild deformations, we would expect the N roots of this equation, which we denote by γ_i', to differ very little from their respective values for the straight guide, γ_i; hence we let

$$\gamma_i'=\gamma_i+\delta_i \tag{15}$$

where $|\delta_i|\ll|\gamma_i|$. With that assumption we can calculate each δ_i as a small perturbation. For mode 1 in particular, we let $\gamma-\gamma_1=\gamma_1'-\gamma_1=\delta_1$, and

$$\gamma_k-\gamma=\gamma_k-\gamma_1'=\gamma_k-\gamma_1-\delta_1\simeq\gamma_k-\gamma_1 \tag{16}$$

which leads to

$$\delta_1=\sum_{k=2}^{N}(C_{1,k})^2(\gamma_k-\gamma_1)^{-1}. \tag{17}$$

This result says that the change in propagation constant of a mode of interest 1 due to a waveguide distortion which couples 1 to modes $2,\cdots,N$, but does not couple the latter to each other, is obtained by summing the contributions due to 1 coupling to the other modes on an individual basis. This derivation does not specify the type of deformation, only that it is mild, and will be used later for circular bends, twists, and certain helical deformations.

Coming back to the three-mode coupling case introduced earlier, we find that the change in attenuation for mode of interest 1, which is the real part of δ_1, is

$$\mathrm{Re}(\delta_1)=\frac{(C_{1,2})^2\Delta\alpha_{2,1}}{(\Delta\alpha_{2,1})^2+(\Delta\beta_{2,1})^2}+\frac{(C_{1,3})^2\Delta\alpha_{3,1}}{(\Delta\alpha_{3,1})^2+(\Delta\beta_{3,1})^2} \tag{18}$$

where

$$\gamma_j=\alpha_j+i\beta_j$$
$$\Delta\alpha_{j,k}=\alpha_j-\alpha_k$$
$$\Delta\beta_{j,k}=\beta_j-\beta_k. \tag{19}$$

Similarly, the change in phase constant is

$$\mathrm{Im}(\delta_1)=\frac{(C_{1,2})^2\Delta\beta_{2,1}}{(\Delta\alpha_{2,1})^2+(\Delta\beta_{2,1})^2}+\frac{(C_{1,3})^2\Delta\beta_{3,1}}{(\Delta\alpha_{3,1})^2+(\Delta\beta_{3,1})^2}. \tag{20}$$

With modes 1, 2, and 3 corresponding to TE_{p0}, $TE_{p+1,0}$, and $TE_{p-1,0}$, respectively, we find, using Table I and the propagation constants in Appendix I, that

$$\alpha'_{p0}=\mathrm{Re}(\gamma'_{p0})\simeq\alpha_{p0}\left[1+A(p)\frac{w^6}{\lambda^4R^2}\right] \tag{21a}$$

where

$$A(p)=\frac{2^7}{\Pi^4}\left[\frac{(p+1)^2}{\left(p+\frac{1}{2}\right)^5}-\frac{(p-1)^2}{\left(p-\frac{1}{2}\right)^5}\right],\qquad p=1,2,3,\cdots. \tag{21b}$$

Equation (21) happens to hold for $p=1$ because the second term of $A(p)$ vanishes and the two-mode coupling expression appropriate for that mode is recovered. Similarly, it is found that

$$\beta'_{p0}=\mathrm{Im}(\alpha'_{p0})\simeq\beta_{p0}\left[1+\frac{p^2A(p)}{8}\frac{w^4}{\lambda^2R^2}\right]. \tag{22}$$

The range of validity of (21a) and (22) corresponds to R being sufficiently large to avoid transition to the whispering gallery regime [5], i.e., $R\geqslant 8w^3/\lambda^2p^2$.

Calculation of a few values of $A(p)$ shows that only TE_{10} experiences an increase in both α and β, whereas all higher order modes experience reductions, the magnitudes of which increase as p gets larger. These conclusions are in qualitative agreement with the work of others [6], [7]. The three-mode coupling analysis provides an insight into the origin of the increase or decrease in loss due to a slight y-bend. TE_{10} is coupled to lossier TE_{20}, and its loss can only go up; on the other hand, for $p\geqslant 2$, TE_{p0} couples to both lossier $TE_{p+1,0}$ and less lossy $TE_{p-1,0}$, in such a way that it experiences an overall decrease in loss.

Equations (21) and (22) can also be quantitatively compared to similar expressions found in [6]. Rewritten in the present notations, these become

TABLE III
COEFFICIENTS $F_xC_{a,b}$ FOR CIRCULAR x-BENDS

b \ a	TE_{pq}	TM_{pq}
$TE_{pq'}$	$\frac{2q^2q'^2}{b^2}+\frac{p^2(q^2+q'^2)}{w^2}$	$\frac{qp(q'^2-q^2)}{wb}$
$TM_{pq'}$	$\frac{q'p(q^2-q'^2)}{wb}$	$qq'\left(\frac{q^2+q'^2}{b^2}+2\frac{p^2}{w^2}\right)$

$$\alpha'_{p0}=\alpha_{p0}\left[1+2B(p)\frac{w^6}{\lambda^4R^2}\right] \tag{23a}$$

$$\beta'_{p0}=\beta_{p0}\left[1+\frac{p^2B(p)}{8}\frac{w^4}{\lambda^2R^2}\right] \tag{23b}$$

where

$$B(p)=\frac{4}{3p^4}\left(\frac{15}{p^2\Pi^2}-1\right),\qquad p=1,2,3,\cdots. \tag{23c}$$

As mentioned earlier, the signs of $A(p)$ and $B(p)$ are the same for all values of p. Further investigation shows that, although these two functions do not look anything like each other, the ratio $A(p)/B(p)$ is within 1.5 percent of unity for all values of p. It follows that the changes in the phase constants given by the two methods (22) and (23b) agree very well; on the other hand, the mode-coupling method predicts a change in losses due to bending, (21a), which is just about half that predicted by the perturbation technique, (23a), which itself agrees with accurate numerical calculations [6]. Interestingly enough, the discrepancy is independent of any physical parameters, including the index of refraction of the wall. This implies that it remains even in the microwave region where the usual expressions for the TE modes, used here to derive the coupling coefficients, should constitute excellent approximations. The origin of the difficulty has been traced to the standard mode-coupling formulation where it is assumed that C is real [1], [3], [12]–[14]. In fact C has a small imaginary part which leads to a doubling of the bending losses, in agreement with the other theories. Since this more accurate calculation requires extensive modifications to the present formalism, it will be presented elsewhere.

The coupling coefficients for circular x-bends can be deduced from those for y-bends by exchanging p with q, p' with q', and w with b. The resulting expressions of $F_xC_{a,b}$ are listed in Table III, where

$$F_x=F_{qp,q'p'}\left(\frac{b}{\lambda R}\right)=\frac{\Pi}{8}(q^2-q'^2)^2\cdot\left(\frac{q^2}{b^2}+\frac{p^2}{w^2}\right)^{1/2}\left(\frac{q'^2}{b^2}+\frac{p'^2}{w^2}\right)^{1/2}\left(\frac{\lambda R}{b}\right). \tag{24}$$

The associated selection rules are that $p=p'$, and that q

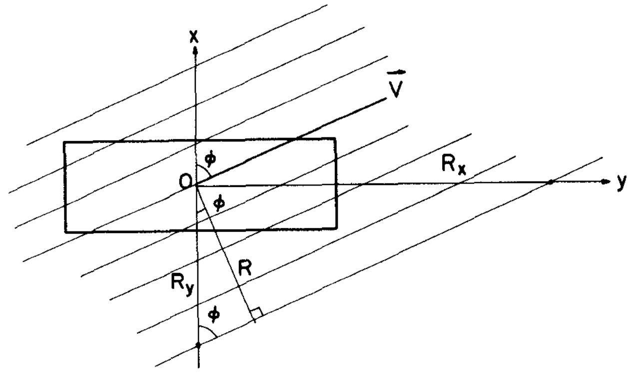

Fig. 3. Geometry for studying bending around an arbitrary direction $\vec{V}$ in the $x-y$ plane. The parallel lines are equiphase lines for f in (25).

and q' must have opposite parity. In particular, these rules indicate that an x-bend will couple TE_{10} to TE_{11}, TE_{13}, etc., as well as to TM_{11}, TM_{13}, etc. Even the first terms in these families, TE_{11} and TM_{11}, are far lossier than the low-order TE_{p0} modes that were excited by a y-bend, and this therefore indicates that an x-bend is going to be far more dangerous for the TE_{10} mode than a y-bend of identical radius.

Consider now a general circular bend around an arbitrary direction in the $x-y$ plane. The coupling coefficients for such a bend can be calculated by the tilt method utilized earlier, but with the exponential term in (4) replaced by

$$f\left(\frac{\delta x}{w}, \frac{\xi y}{b}\right) = \exp\left[\frac{i2\Pi}{\lambda}(\delta x + \xi y)\right] \tag{25}$$

which indicates that the bend (tilt) considered is around the direction parallel to the vector $\vec{V}$ of components $V_x = \xi$ and $V_y = -\delta$. The coupling coefficients are then obtained from (3), and the coupling coefficients per unit length of bend by

$$C_{a,b} = \lim_{\delta,\xi,l\to 0}\left[l^{-1}K_{a,b}(\delta,\xi)\right] \tag{26}$$

the limit being taken subject to the conditions

$$\frac{l}{\delta} = R_y = \frac{R}{\cos\phi} \quad \text{and} \quad \frac{l}{\xi} = R_x = \frac{R}{\sin\phi} \tag{27}$$

where R is the total radius of the bend around $\vec{V}$ with the x-axis (Fig. 3). For $a \neq b$, $C_{a,b}$ is of order R^{-1}. Expanding f to the same order and carrying out the calculation of $C_{a,b}$, it is found that the latter is simply the sum of two separate contributions, respectively, due to a y-bend of radius R_y and an x-bend of radius R_x. Since the selection rules for x- and y-bends are mutually exclusive, it is clear that the x and y components of a general circular bend respectively couple any TE_{pq} or TM_{pq} mode to two distinct, nonoverlapping, families of modes. Hence, each coupling coefficient is either a coefficient obtained from Table I, or from Table III, or it vanishes. Thus, one really never has to add two different coefficients, but merely to choose the appropriate coefficient in the tables. This implies that when calculating the extra loss of a TE_{pq} or TM_{pq} mode due to a general circular bend, as obtained from (17), we find

$$\begin{aligned}\mathrm{Re}(\delta_{pq}) &= A_{pq}(R_x)^{-2} + B_{pq}(R_y)^{-2} \\ &= \left(A_{pq}\cos^2\phi + B_{pq}\sin^2\phi\right)R^{-2}\end{aligned} \tag{28}$$

where A_{pq} and B_{pq} are constants which depend on the mode under study. Even though it is, in general, difficult to calculate these coefficients explicitly, some simple conclusions can be drawn from the form of (28). In particular, we can compare the effect of ϕ upon bends of given R. It is clear that if $A_{pq} < B_{pq}$, the least attenuated bend will be obtained for $\phi = 0$, i.e., for a y-bend; conversely an x-bend will yield minimum attenuation if $A_{pq} > B_{pq}$. If $A_{pq} = B_{pq}$, the excess loss will be independent of ϕ. In the case of TE_{10}, A_{10} results primarily from coupling to TE_{20}, whereas B_{10} is mostly due to coupling to the lossier modes TE_{11} and TM_{11}, so that $B_{10} > A_{10}$, and a y-bend is necessary to minimize the losses for a given bend radius R. This is a fortunate result since rectangular metallic waveguides designed for low-loss transmission of infrared radiation have $b \gg w$ and thus strongly resist being bent around the x-axis for mechanical reasons. This lack of flexibility in one direction, which might be viewed as a hindrance from a purely mechanical standpoint, thus turns out to be a blessing in disguise: it provides a natural resistance to the higher losses which would otherwise be associated with x-bends.

III. Twists

Another situation where modes are coupled with constant coupling coefficient per unit length is the case where the z-axis of the guide remains straight, but its cross section rotates at a constant angular rate per unit length τ along that axis, giving the guide the appearance of a corkscrew; we refer to this deformation as a twist. We can calculate the coupling coefficients per unit length of twist by approximating this structure by a series of straight sections of length l rotated by a small angle ϵ with respect to each other, calculating the coupling coefficient $K'_{a,b}(\epsilon)$ between two such sections from (3) and taking the appropriate limit subject to $\epsilon/l = \tau$. Specifically, the field $\vec{E}'_a$ of mode a propagating in one section is related to the field of the same mode in the next section $\vec{E}_a$ by a rotation of angle ϵ around the z-axis (Fig. 4). In other words

$$\begin{bmatrix}(E'_a)_x \\ (E'_a)_y\end{bmatrix} = \begin{bmatrix}\cos\epsilon & \sin\epsilon \\ -\sin\epsilon & \cos\epsilon\end{bmatrix}\begin{bmatrix}(E_a)_x \\ (E_a)_y\end{bmatrix}. \tag{29}$$

Hence

$$\begin{aligned}\iint \vec{E}'_a\cdot\vec{E}_b\,dx\,dy &= \cos\epsilon\iint\left[(E_a)_x(E_b)_x + (E_a)_y(E_b)_y\right]dx\,dy \\ &\quad + \sin\epsilon\iint\left[(E_a)_x(E_b)_y - (E_a)_y(E_b)_x\right]dx\,dy \\ &\simeq \epsilon\iint\left[(E_a)_x(E_b)_y - (E_a)_y(E_b)_x\right]dx\,dy\end{aligned} \tag{30}$$

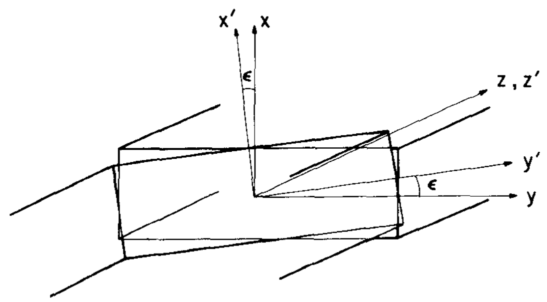

Fig 4. Geometry to calculate the coupling coefficients resulting from a twist around the z-axis. The primed and unprimed coordinate systems correspond to the waveguide sections before and after the discontinuity, respectively.

TABLE IV
COEFFICIENTS $LC_{a,b}$ FOR TWISTS AROUND THE z-AXIS

b \ a	TE_{pq}	TM_{pq}
$TE_{p'q'}$	$p^2q'^2 - q^2p'^2$	$\left(\frac{p'^2}{w^2} + \frac{q'^2}{b^2}\right)pqwb$
$TM_{p'q'}$	$\left(\frac{p^2}{w^2} + \frac{q^2}{b^2}\right)p'q'wb$	0

since the coefficient of $\cos\epsilon$ vanishes by orthogonality of modes a and b (we do not calculate the self-coupling coefficient here, i.e., $a \neq b$). Using (30) in (3), and the appropriate expressions from the Appendix, the coupling coefficients per unit length are obtained by

$$C_{a,b} = \lim_{\epsilon, l \to 0} \left[l^{-1} K'_{a,b}(\epsilon) \right], \qquad \epsilon / l = \tau. \tag{31}$$

The results are listed in Table IV in the form of the product $LC_{a,b}$, where the function L is defined by

$$L = L_{pq,p'q'}(\tau) = \frac{\Pi^2}{16}\left(\frac{p^2}{w^2} + \frac{q^2}{b^2}\right)\left(\frac{p'^2}{w^2} + \frac{q'^2}{b^2}\right) \cdot (p^2 - p'^2)(q'^2 - q^2)\frac{wb}{\tau}. \tag{32}$$

The selection rules are that p and p' must be of opposite parity, and that q and q' must also be of opposite parity. These rules are not as restrictive as in the case of circular bends, and as a result it is difficult to look at this situation as a two- or three-mode coupling scheme because of the large number of TE and TM modes being coupled to. Even for TE_{10} it is difficult to ascertain which modes make the largest contributions to the increase in loss.

We can nevertheless draw some qualitative conclusions about twists. Since all the coupling coefficients are proportional to τ, (17) shows that the increase in loss due to a twist is proportional to τ^2; this quadratic behavior has been observed in practice [15]. A general feature of mode-coupling theory is that in the limit of weak coupling the eigenmodes of the perturbed guide are very similar in structure to those of the straight guide. For a twisted rectangular guide, this means that the eigenmode close to TE_{10} will be very similar to the latter, with the electric vector nearly parallel to the long side of the guide cross section at any location along z; in other words, the direction of the electric field will rotate at the same rate as the guide. This property has also been observed experimentally [15].

IV. HELICAL DEFORMATIONS

Thus far we have considered separately circular bends and twists which may occur in the course of the utilization of flexible rectangular metallic waveguides for the infrared. We now study deformations which are combinations of circular bends and twists, namely helical deformations, for which the mode-coupling coefficients are simply the sum of those attached to the elementary deformations [16], which have been calculated in the preceding sections. For a guide with $b \gg w$, we have seen that x-bends are undesirable, and also difficult to obtain in practice, and we therefore now restrict the discussion to structures obtained with y-bends and twists only, which we refer to as a natural helical deformation (Fig. 5).

For such a helical structure, then, the mode-coupling coefficients $C_{a,b}$ are obtained by summing the elements of Tables I and IV, where R and τ now take on the significance of the radius of curvature and the torsion of the helix, respectively. These are related to the parameters d and e of the helix (Fig. 5) by

$$R = \frac{d^2 + e^2}{d} \quad \text{and} \quad \tau = \frac{e}{d^2 + e^2}. \tag{33}$$

Since the selection rules associated with Tables I and IV are incompatible, the coupling coefficient due to a helical deformation for any mode a to any other mode $b \neq a$ can simply be read from one of these tables, and is never the sum of two nonvanishing terms, one from each table. According to (17) and this remark, then, the increase in loss $\operatorname{Re}(\delta_1)$ of any mode 1 due to a helical deformation can be put in the form

$$\operatorname{Re}(\delta_1) = CR^{-2} + D\tau^2 = CR^{-2}\left[1 + E(R\tau)^2\right] \tag{34}$$

where C, D, and $E = D/C$ are constants related to the characteristics of the coupled modes. Thus we conclude that, for a natural helical deformation, the loss increase is the sum of terms proportional to the squares of the curvature and the torsion, in agreement with the dependence proposed by other workers [8]. The form of this equation is also interesting in that it says that the excess loss of the helical guide is that of the same guide with a circular bend only (CR^{-2}), times a factor of the form $1 + E(R\tau)^2$. This statement is virtually identical to that made for helical whispering gallery guides in a study where the significance of the dimensionless quantity $R\tau$ was brought out [17], and where it was found that $E = 1$ for TE polarized waves in metallic guides in the infrared. Due to the large number of modes involved in the mode-coupling description, and to the uncertain expressions for their losses, it is not possible to derive similarly simple expressions in the present context, even for TE_{10}.

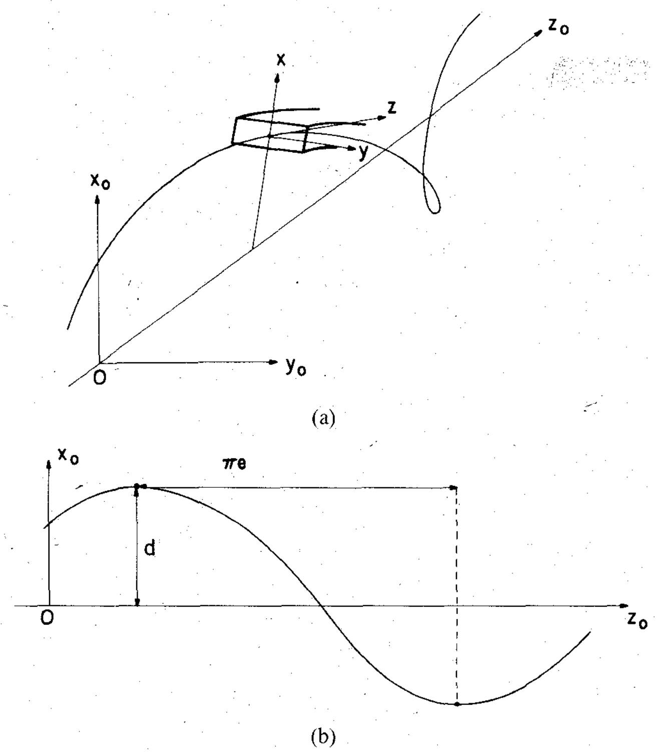

Fig. 5. Natural helical bend of a rectangular guide. (a) General view; the z_0-axis is along the axis of the helix. (b) Side view of the helix, defining d and e.

V. Discussion

The influence of helical deformations upon the losses of the eigenmodes of rectangular metallic waveguides has been studied by means of multimode coupling theory. Such deformations result from the combination of elementary rotations about the local x, y, and z axes of the guide. The coupling coefficients between all propagating straight guide modes due to these individual orthogonal rotations have been calculated, and the corresponding selection rules indicate that the following remarkable properties hold.

1) Any particular mode a is coupled by the three types of orthogonal rotations to three distinct, nonoverlapping, families of modes b. Thus, in the case of an arbitrary helical deformation, the coupling coefficient between two particular modes a and b is not a linear combination of three nonzero coupling coefficients, but reduces to just one of them.

2) If a mode a is coupled to two modes b and c by an elementary rotation about one of the three axes, then b and c themselves are not coupled by that rotation. This property is also valid for coupling by deformations composed of elementary rotations about two orthogonal axes, as encountered in general circular bends and natural helical deformations. It can readily be seen by comparing the selection rules, however, that this does not hold for deformations made up of all three elementary rotations. For general helical deformations, then, one could not use (12)–(17).

When applicable, these properties greatly facilitate the calculation of excess loss of modes due to helical deformations. Property 2) shows that we are then in the conditions of applicability of the simple multimode coupling theory developed here to deal with situations which cannot be adequately covered by the usual two-mode coupling theory, so that the excess loss can simply be obtained from (17). Property 1) then simplifies this calculation since the various coupling coefficients entering (17) can simply be read from Tables I, III, or IV without further combination. This method can lead to expressions for the excess loss whenever accurate formulas exist for the losses of the principal modes being coupled by the deformation. This is the case for TE_{p0} modes in a guide deformed by a y-bend only, and we have derived approximate analytic expressions for the excess loss in this situation and compared them to other published expressions. We could not, however, carry out similarly detailed calculations for natural helical deformations, because of lack of accuracy of the attenuation constants of some of the prominent modes; progress in this direction will thus require refinements of the calculations of the attenuation constants of TE_{pq} $(q>0)$ and TM_{pq} modes of straight rectangular infrared guides. Nevertheless, properties 1) and 2) allowed us to study qualitatively the excess loss under natural helical deformations and general circular bending; in the latter case it was found that, for typical infrared waveguide configurations, minimum excess loss is fortunately obtained by mechanically favored deformations.

This study furthers the understanding of the effect of deformations on the losses of multimode rectangular metallic waveguides. Although the primary incentive for this work was the development of hollow guides for CO_2 laser radiation, the results presented here should find applications throughout the rapidly developing mid-infrared, far-infrared, and submillimeter regions of the electromagnetic spectrum.

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Appendix

Transverse electric fields in rectangular guides:

TE_{pq}

$$E_x=\frac{q\Pi}{b}\cos\frac{p\Pi x}{w}\sin\frac{q\Pi y}{b},\qquad p=0,1,2,\cdots,$$

$$E_y=-\frac{p\Pi}{w}\sin\frac{p\Pi x}{w}\cos\frac{q\Pi y}{b},\qquad q=0,1,2,\cdots,$$

$$p\neq q=0. \tag{35}$$

TM_{pq}

$$E_x=\frac{p\Pi}{w}\cos\frac{p\Pi x}{w}\sin\frac{q\Pi y}{b},\qquad p=1,2,\cdots,$$

$$E_y=\frac{q\Pi}{b}\sin\frac{p\Pi x}{w}\cos\frac{q\Pi y}{b},\qquad q=1,2,\cdots. \tag{36}$$

Useful integrals (low-order expansions in powers of $u = \sigma a/\lambda$):

$n \neq m$

$$I(n,m,a,\sigma) = \int_0^a \sin\frac{n\Pi t}{a}\sin\frac{m\Pi t}{a}e^{\frac{i2\Pi\sigma t}{\lambda}}dt \qquad (37)$$

$$= \frac{8a}{\Pi i}\frac{nm}{(n^2-m^2)^2}u, \qquad \text{if } n, m \text{ of opposite parity}$$

$$= \mathcal{O}(u^2), \qquad \text{if } n, m \text{ of same parity}$$

$$J(n,m,a,\sigma) = \int_0^a \cos\frac{n\Pi t}{a}\cos\frac{m\Pi t}{a}e^{\frac{i2\Pi\sigma t}{\lambda}}dt \qquad (38)$$

$$= \frac{4a}{\Pi i}\frac{n^2+m^2}{(n^2-m^2)^2}u, \qquad \text{if } n, m \text{ of opposite parity}$$

$$= \mathcal{O}(u^2), \qquad \text{if } n, m \text{ of same parity.}$$

$n = m$

$$I(n,n,a,\sigma) = \frac{a}{2}\frac{e^{i2\Pi u}-1}{i2\Pi u}\frac{n^2}{n^2-u^2}$$

$$\simeq \frac{a}{2}\left[1+\left(\frac{1}{n^2}-\frac{2\Pi^2}{3}\right)u^2+i\Pi u\right]. \qquad (39)$$

Propagation constants ($b, w \gg \lambda$):

TE_{p0}

$$\gamma_{p0} = \alpha_{p0} + i\beta_{p0} = \frac{p^2\lambda^2}{2w^3}\operatorname{Re}(\nu^{-1}) + i\left(\beta_0 - \frac{\Pi p^2\lambda}{4w^2}\right) \qquad (40)$$

where ν is the index of refraction of the wall, $\beta_0 = 2\Pi/\lambda$, and λ is the freespace wavelength.

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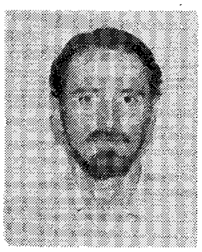

Michel E. Marhic (M'79) was born in France in 1945. He received the Diplome d'Ingenieur degree from the Ecole Superieure d'Electricite in 1968. He then obtained the M.S. degree from Case Western Reserve University, Cleveland, OH, in 1970, and the Ph.D. degree from the University of California at Los Angeles in 1974, both in electrical engineering.

He has been with Northwestern University, Evanston, IL, since 1974, where he is presently Associate Professor of electrical engineering. He spent the 1979–1980 academic year on sabbatical at the University of Southern California in Los Angeles. His research interests are in the area of laser applications, including the transmission of high power through hollow waveguides.